

Second-Order Scaling in the Two-Flavor QCD Chiral Transition

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Abstract

Scaling behavior is analyzed for the two-flavor QCD lattice gauge theory chiral transition. Leading scaling behavior and correction to leading scaling from lattice spacing effects are examined for the quark condensate. Scaling predictions under the assumption of quark mass dominance are tested for the longitudinal correlation length. Second order scaling behavior is consistent with present data.

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I. Introduction

Some time back it was pointed out by Pisarski and Wilczek [1], that the QCD chiral phase transition has a suitable order parameter from which an effective, Landau-Ginzburg-Wilson, critical theory could be constructed. Recently Wilczek [2] has examined the case of two fermion flavors in the chiral limit, where the effective LGW-theory is now the $SU(2) \times SU(2) = O(4)$ Heisenberg model. This latter model has been well studied for spin systems in statistical mechanics. As such, the necessary information regarding the critical behavior of this theory was immediately available. Subsequently Rajagopal and Wilczek [3] gave a transcription dictionary which related the $O(4)$ -Heisenberg transition into the language of lattice gauge theory. In particular the quark mass and quark condensate transcribed to the magnetic field and magnetization respectively in the language of spin systems. This transcription now readily allows testing the original assertion, that the chiral phase transition is second order, by checking for universal scaling behavior and comparing it to that of the $O(4)$ -Heisenberg model.

As lattice simulation data on the $SU(2)$ -chiral transition increases, it will become possible to analyze the scaling behavior near the tentative transition point. Such tests will provide important input in establishing the correctness of the above hypothesis by [2] that the transition is second order and in the $O(4)$ -Heisenberg universality class. To obtain reliable estimates from such analyses, a good understanding is first needed of the relevant scales involved in real finite temperature simulation studies. In general there are four scales involved here, which in descending order are the length of the lattice box L , the correlation length ξ , the QCD scale Λ_{QCD} , and the lattice spacing a . Ideally, for studying critical phenomena scaling, one would like $L \gg \xi \gg \Lambda_{QCD} \gg a$. In real simulation, typical hadron sizes span only a few lattice spacings, and only two or three hadrons fit inside the lattice volume. This situation places greater problems in studies of critical phenomena, above those well familiar from zero temperature studies. Firstly the divergence of the correlation length is impeded by the lattice volume, leading to finite size effects. In addition, since asymptotic scaling in the ultraviolet is only approximate, hadron states exhibit a -dependence, which in turn complicates infrared critical phenomena scaling through the addition of correction terms [4]. Furthermore, the two problems are complementary in a fixed lattice volume. For, to improve asymptotic scaling by decreasing the physical size of a lattice spacing, (ie placing more lattice sites inside a hadron state), goes at the expense of decreasing the maximum physical size that the correlation length can achieve.

To overcome such difficulties, there is some consideration [5] that it is more advantageous to examine critical phenomena scaling at finite, not vanishing, quark mass. In such a case, for a sufficiently large quark mass, scaling theory tells us that the correlation length behaves as**

$$\xi = \frac{A}{h^{\nu/\Delta}} (1 + a_1(t/h^{1/\Delta}) + \dots) , \quad (1.1)$$

so is cutoff by a scale set by the quark mass. Here Δ is the gap exponent, related by the scaling laws to δ and β by $\Delta = \beta\delta$. For the $O(4)$ system $\beta = 0.38 \pm 0.01$, $\delta = 4.82 \pm 0.05$ [6] so that $\Delta = 1.83 \pm 0.06$.

However, the more one suppresses the correlation length, the more significant are the effects from the microscopic scales. The dominant contributions are from dimension four operator corrections arising from the microscopic details of the system. In particular, these corrections contain the dominant effect of the nonzero lattice spacing.

Let us estimate the size of this effect on the magnetization. The scaling form for the magnetization is [7]

$$M = Bh^{1/\delta} f(|t|/h^\Delta) (1 + b_1|t|^{\omega\nu} + \dots) , \quad (1.2)$$

and at $t=0$ it is

$$M = Ch^{1/\delta}(1 + c_1 h^{\frac{\omega\nu}{\Delta}} + \dots) . \quad (1.3)$$

The second term inside the parenthesis in both equations above, is the correction to leading scaling term arising from a nonzero lattice spacing. Here, for the O(4) system, $\nu = 0.73 \pm 0.02$ [6] and $\omega = 0.46$ [8, 9, 10]. Away from $t = 0$, when $t/h^{\nu/\Delta} \gg 1$ recall $\xi \sim t^{-\nu}$. At $t = 0$, the magnetic field will dominate, where from Eq. (1.1) we find $\xi \sim 1/h^{\nu/\Delta}$. In both cases the correction to scaling term goes as $\xi^{-\omega}$. Let us suppose that the correlation length is ten lattice units long, a size much bigger than found in simulations. If we assume that the coefficients C and $C * c_1$ for the leading and next-to-leading scaling terms respectively in Eq. (1.3) are of about the same magnitude, then the latter term gives about a 30 percent correction.

In this paper we will study scaling behavior. Due to limited data from lattice gauge simulations, we can not call this a legitimate scaling analysis. A legitimate scaling analysis would require demonstrating a consistent fit of many data points to expected scaling forms. At present we are working with a hopeless three or four points in any single type of scaling analysis. However, there are a few qualitative and also quantitative predictions from scaling theory that can be tested with sparse data. One such qualitative prediction is the correction to leading scaling from effects of a nonzero lattice spacing. Lattice spacing effects have frequently been cited in various studies [11, 12, 13]. The presence of a nonzero lattice spacing necessarily implies that there will be corrections to leading scaling. The theory of second order critical phenomena predicts well prescribed functional forms for these correction terms. In section 2, under the assumption that the chiral transition is second-order, we will examine those corrections to leading scaling behavior in the magnetization, which arise from a nonzero lattice spacing. Quantitatively, scaling behavior of the longitudinal correlation length is a key indicator and can be tested with present data. In section 3 we will make estimates of its scaling behavior based on lattice simulation data.

II. Magnetization

Lattice simulators have been examining the magnetization, $\langle \bar{\psi}\psi \rangle$, for some time now. There appears sufficient consistency in their results to warrant closer analysis. In figure (1) the plotted points are from simulation data in [14, 15, 16]. The solid line is a fit to Eq. (1.3) without the leading scaling correction term. The dashed curve is a fit with the correction term added. Both are for the O(4) exponents. A similar fit to O(2) exponents gives minute differences, undetectable on the graph.

The results we present for the magnetization demonstrate the effect of the lattice spacing in the scaling forms, when everything else is kept the same. Rather than extrapolating a zero magnetic field critical temperature, we allowed for the analytic shifts that arise in the presence of a magnetic field. The exact value of these transition points would require outside nonperturbative information which is not calculable at present. In our fitting procedure, we simply estimated the transition point at each m_q , as the place of maximum drop-off in the magnetization profile as a function of β . Let us justify the sufficiency of this crude method for present purposes. In a more rigorous treatment one may make a best fit with the critical temperature as an initial unknown fitting parameter. A suggestive form for such a fit would be the leading expansion to Eq. (1.2),

$$M = Ah^{(1/\delta)}(1 + a_1 |t|/h^{1/\Delta}) . \quad (2.1)$$

By this approach, one would be committed to a single critical temperature for all values of m_q . One may worry about analytic corrections to β_c as a function of m_q . This concern may be further extenuated here since our "temperature" parameter. $\beta = 6/g^2$ is itself analytically related to the physical temperature parameter. From present data, indications are that the variation of β_c is only

about 2 percent for quark masses ranging from 0.004 to 0.0125. To be precise, $\beta_c=5.48, 5.475,$ and 5.54 for $m_q=0.004, 0.00625$ and 0.0125 respectively. The outer two are given in [15] and [14] respectively and the middle point is from our estimate. At a later stage, when more data points are available, it may make sense to use the form (2.1) at each m_q so as to also make $\beta_c(m_q)$ a fitting parameter.

One point which becomes clear from the above examination of the magnetization is that it will be difficult to differentiate between $O(2)$ and $O(4)$ critical behavior even with increased amount of data. This is because lattice correction effects dominate. A tractable question that one can hopefully decide upon is whether the transition is second order. Here general properties of scaling forms, if consistently matched with data, will be a significant advance. In particular one can pose the question in a simplified fashion for present needs as deciding whether the transition is first or second order or whether it is some sort of smooth transition. Application of scaling theory should be sufficient to decide between the first two options. Establishing that the longitudinal correlation length diverges should place considerable doubts on the latter possibility.

III. Longitudinal Correlation Length

Let us now turn to the longitudinal correlation length, which in the lattice gauge theory language is the inverse sigma screening mass, $1/m_\sigma$. There are concerns [17] that for m_σ only the isospin $I=1$ channel contributed in the lattice gauge measurements of [14, 18]. This, in particular, means that the ground state $I=0$ sector is not contributing in the measurement of m_σ . For zero temperature properties in the scalar sector, this elimination of the $I=0$ sector would be a cause for obvious concern. Near the critical point, one expects scalar particles from all low lying isospin sectors to contribute with equal importance. Under this assumption we expect the universal scaling properties of critical behavior to still be reflected in m_σ . We follow this reasoning and assume in the analysis below that $\frac{1}{m_\sigma}$, as measured by lattice simulations in [14, 18], is the longitudinal correlation length.

A qualitative assessment of the longitudinal correlation length in [14] shows that it increases a little as one approaches β_c from below. In the case of the hot start, it peaks near β_c . It then drops a bit as one goes to higher β and finally levels off. In the cold start case, although there is no peaking, there is still a definite rise as it approaches β_c and then a leveling off.

In a system nearing a second order phase transition, if the diverging correlation length is influenced by the box size, it is seen by its sharp rise being cutoff at some intermediate value. In this region the finite extent of the box modifies its behavior to [19],

$$\xi = BL(1 + b_1|t|L^{1/\nu} + \dots) . \quad (3.1)$$

If in the same system, a large enough magnetic field is imposed, the correlation length is also flat near β_c , as seen from Eq. (1.1). Also for comparison, if the temperature dominates over the volume and quark mass, the simplest form consistent with scaling theory is,

$$\xi = a|t|^{-\nu} . \quad (3.2)$$

Using the data in [14], which is fairly representative of lattice simulation data for m_σ , the divergence of the longitudinal correlation length evidently is cutoff as it approaches β_c . We anticipate that either the volume or quark mass controls the correlation length near β_c . We will define a region as quark mass (volume) dominated, when the divergence of the correlation length as $t \rightarrow 0$, is stopped due to the presence of a quark mass (a finite volume). Thus in the quark mass dominated region, for example, if one lowers the quark mass, it will increase ξ as $t \rightarrow 0$. However, if one increases the volume and keeps the quark mass fixed, ξ will not increase as $t \rightarrow 0$.

To judge between regions of volume and quark mass dominance, we first observe from Eq. (1.1) that

$$\frac{\xi(2m_q)}{\xi(m_q)} = \frac{1}{2^{\nu/\Delta}} = 1/2^{0.4} = 0.76 , \quad (3.3)$$

where the exponent used on the right hand side is for the O(4) system. There is some data at two values of quark mass with fixed box size $L=12$ and $N_t = 6$ in [18]. Using the critical temperatures they estimated from the magnetization profiles, $\beta_c(m_q = 0.0125) = 5.42$ and $\beta_c(m_q = 0.025) = 5.445$, we find the ratio of the measured correlation lengths at the respective critical points to be,

$$\frac{\xi_m(0.025)}{\xi_m(0.125)} = \frac{0.65 \pm 0.03}{0.839 \pm 0.009} = 0.77 \pm 0.05 , \quad (3.4)$$

which agrees with eqs (3.3). Above the subscript m is to indicate measured value. The closeness of the two ratios in Eqs. (3.3) and (3.4) should not be taken too seriously considering the approximations involved. Nevertheless, from a qualitative inspection of the data in the transition region, it is evident that the correlation length at $m_q = 0.0125$ is larger by roughly the above amount. This suggests that down to $m_q = 0.0125$ the quark mass effects dominate the finite size effects at $L=12$. This offers numerical support to one of the assumption on which the recent analysis by Karsch [5] is based.

We can also make a comparison of $\xi(\beta_c)$ at $m_q = 0.0125$ for lattice sizes of $L=12$ and 16 from the data in [18] and [14] respectively. The former is for $N_t = 6$ and the latter is for $N_t = 8$. From this comparison, one can assess the degree to which the continuum limit (i.e. ultraviolet scaling limit, where the theory becomes independent of its short distance cutoff) has been reached near the transition point. To understand our reasoning here, first recall that the critical point in both systems corresponds to the same physical value for the temperature. This implies that the physical scale of a lattice unit in the two systems, as given by $a_{N_t} = \frac{1}{N_t T_c}$, is different. Since our considerations below only involve comparing two lattice systems, for convenience let us define the lattice spacing in the $N_t = 6$, $L=12$ system to be unity and refer to this as the physical unit of length. Then in these units, if the continuum limit has been reached, the $N_t = 8$, $L=16$ system should reproduce the same physics when one sets the lattice spacing at $3/4$. In particular the system size in lattice units of the $N_t = 8$ system, L_8 , is in physical units

$$L_8^{ph} = \frac{3}{4} L_8 . \quad (3.5)$$

For the two systems under comparison, this implies their physical sizes are the same. Thus we can not study finite size scaling effects in comparing these two systems.

Turning to the correlation length, define $\xi_8(t)$ ($\xi_6(t)$) as the correlation length in the $N_t = 8$ ($N_t = 6$) system in units of its lattice spacing. If the continuum limit has been reached, at zero quark mass and at the same physical temperature, we expect to find from measurement the relation,

$$\xi_6(t) = \frac{3}{4} \xi_8(t) , \quad (3.6)$$

where t is the reduced temperature. If, as in our case, the same dimensionless magnetic field (quark mass) is applied to the two systems, the above relation is modified. One can intuitively see this by noting that if the same dimensionless magnetic field is applied on each lattice site, then the system with the greater thickness (the $N_t = 8$ system) feels a greater overall effect. We can see this numerically by accounting for dimensions. One has,

$$m_q^{ph} = \frac{m_q}{a} , \quad (3.7)$$

where a is the lattice spacing, the superscript ph implies the quark mass in physical units, and m_q is the dimensionless quark mass. Recalling that we are considering the lattice spacing in the $N_t = 6$ system as our physical length unit, we imagine fitting the parameters in Eq.(1.1) (ie A, a_1 etc...) in these units. Let us now analyze how well the quark mass is controlling the correlation length in the two systems. From Eq. (1.1) we expect,

$$\frac{\xi_8^{ph}(0)}{\xi_6^{ph}(0)} = \left(\frac{m_6^{ph}}{m_8^{ph}}\right)^{\frac{\nu}{\Delta}} = (3/4)^{0.399} = 0.89 . \quad (3.8)$$

From simulation data in [14] we find at the critical point that ^{***},

$$\xi_{8m}^{ph}(0) = \frac{3}{4}\xi_{8m}(0) = (3/4)\frac{1}{0.561 \pm 0.005} , \quad (3.9)$$

where the subscript m is to indicate the measured value. For the measured ratio at the critical point, it is,

$$\frac{\xi_{8m}^{ph}(0)}{\xi_{6m}^{ph}(0)} = \left(\frac{3}{4}\right)\left(\frac{0.65 \pm 0.03}{0.561 \pm 0.005}\right) = 0.869 \pm 0.041 . \quad (3.10)$$

The agreement between eqs. (3.8) and (3.10) offers plausible support for the joint assumption of quark mass dominance and continuum behavior. This single comparison is insufficient to make any conclusions, but it serves as an indicator based on the available data.

IV. Conclusion

The rise in the measured magnetization, as shown in figure 1, is too sharp to fit leading scaling expectations. We are able to account for its behavior by a parametric fit to a scaling form that includes effects of a nonzero lattice spacing. Of course, to fit three data points with two parameters is no feat and at present that is all we are able to do. Nevertheless, we still learn from this analysis that leading scaling behavior is probably insufficient. Also, by turning to outside physics considerations as discussed in [4] and finding support from our curve fit, it tentatively appears that corrections to leading scaling from a nonzero lattice spacing may be the explanation.

Turning to the issue of quark mass dominance, in [20] a relation was given that separates the regimes in which the quark mass and the lattice size control the correlation length. From our findings, we can fit this equation to obtain an upper limit on the smallest quark mass, which separates it from the volume dominated region. In lattice units of the $N_t = 6$ system, we get

$$m_q > CL^{-b} = 6.1L^{-2.49} , \quad (4.1)$$

where $b = \frac{\gamma\delta}{\nu(\delta-1)} = 2.49$.

To the limited extent of our analysis, the longitudinal correlation length apparently exhibits behavior characteristic to a second order phase transition. It should be monitored in future lattice gauge simulations. At present, the HEMCGC collaboration [16] is computing m_σ at $m_q = 0.00625$ for a $16^3 \times 8$ lattice. If we continue to accept quark mass dominance and continuum behavior, similar to Eq.(3.3) we expect from Eq. (1.1),

$$\frac{\xi(m_q = 0.0125, t = 0)}{\xi(m_q = 0.00625, t = 0)} = \frac{1}{2^{\nu/\Delta}} = 0.76 . \quad (4.2)$$

Should this fail, the first explanation that we can offer is a crossover from the quark mass dominated to the volume dominated regime. In the event of such a breakdown, the result would still be useful

in obtaining a more definitive boundary in Eq. (4.1). Before doing this, however, should Eq. (4.2) not be confirmed by measurement, larger lattices would have to be tried in order to be sure that the cause is volume effects. If we rescale Eq. (4.1) for the $N_t = 8$ case, we expect that for $m_q = 0.00625$ the mass dominated region should occur for $L > 19$, where this is in lattice units of an $N_t = 8$ system. For lattices bigger then this, we minimally expect Eq. (4.2) to hold. If so, then one can reasonably assume that simulation data is exhibiting the simplest form of second order scaling. This would be a marked advantage for lattice theorists in further exploration of this region. If the outcome is not in line with the above expectations, then the situation will need more consideration, and the analysis of section 3 may be a fortuitous coincident, but otherwise hollow without content.

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** This relation can be obtained by standard scaling arguments as follows. We first write the asymptotic behavior of the correlation length at zero field in the familiar form $\xi \sim t^{-\nu}$ [21]. In the presence of a finite magnetic field h , although the nonanalytic behavior in t will remain the same, ξ in general can depend on h also. We can write this as $\xi \sim t^{-\nu} g'(t, h)$. By the homogeneity hypothesis of scaling, the dependence in fact can only be with respect to the ratio $\frac{t}{h^\sigma}$. This implies the general form $\xi \sim t^{-\nu} g(\frac{t}{h^\sigma})$, where $g(x)$ is of degree γ . The ratio $\frac{t}{h^\sigma}$ must be the same as in the singular part of the free energy. For the free energy it is well known that $\sigma = 1/\Delta$ [21].

In the region $t=0$, $h \neq 0$, ξ is finite with a nonanalyticity at $h=0$. In order to satisfy this requirement, we must have $t^{-\nu} (\frac{t}{h^{1/\Delta}})^\gamma = \frac{1}{h^{\sigma'}}$. This relation implies $\gamma = \nu$ so that $\sigma' = \frac{\nu}{\Delta}$. Extracting this leading singular piece from g , we can write it as, $g(t/h^{1/\Delta}) = (t/h^{\nu/\Delta}) f(t/h^{1/\Delta})$, where $f(x)$ is analytic in x . From this we obtain eq. (1).

*** The chiral restoration point is quoted in [14] to lie within an interval between two measured points. We took the midpoint in this interval to be the critical point and computed the σ -screening mass by linear interpolation. For the high temperature point of this interval, both a hot start and cold start result were given for m_σ . We used the hot start result in Eq. (3.10). The cold start result changes the ratio R in Eq. (3.10) to $R = 0.844 \pm 0.040$. Also if we use the values at the ends of the interval, we find using the value at the lower end, $\beta = 5.525$, the ratio $R = 0.813 \pm 0.043$. At the upper end, $\beta = 5.55$, we find $R = 0.934 \pm 0.044$ for the hot start and $R = 0.878 \pm 0.43$ for the cold start.

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Figure Caption:

figure 1: Magnetization (quark condensate, $\langle \bar{\psi}\psi \rangle$) versus magnetic field (dimensionless quark mass, m_q). The plotted points are from lattice gauge measurements, the solid line is a fit to the leading scaling form, and the dashed line is a fit with the first correction term to leading scaling added.

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